

# Lifshitz-like argument for low-lying states in a strong magnetic field

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**Abstract.** Localization of an electron moving in two dimensions, submitted to a strong magnetic field and scattered by randomly distributed zero-range impurities is investigated. Considering the explicit expression for the density of states obtained by Brézin, Gross and Itzykson, the Lifshitz argument is adapted in order to analyze the unusual power-law behavior of the low energy spectrum. When the impurity density is smaller than the Landau degeneracy, typical configurations of disorder responsible for low energy states are identified as cluster of impurities of well defined form. This allows for an interpretation of low-lying states, localized around these clusters, whose size diverges logarithmically as the energy goes to zero.

**PACS.** 73.40.Hm Quantum Hall effect (integer and fractional) – 72.15.Rn Localization effects (Anderson or weak localization) – 05.30.-d Quantum statistical mechanics

## 1 Introduction

The problem of an electron submitted to a strong magnetic field and moving on the plane in a random potential, has been subject of intensive investigations, due to its relevance for the integer quantum Hall effect. In the case of a locally correlated disordered potential, explicit results have been found regarding the average density of states (DOS) [1–3]. Although the DOS does not contain in general any information about localization, exception should be made for the tails of the spectrum, which are generally associated with improbable realizations of the random potential. And as for the Lifshitz-tail examples, these tails are interpreted in terms of localized states. In the strong magnetic field problem, such a situation is encountered for the case of Gaussian fluctuations (the spectrum displays a Gaussian tail at large energy [1]). Things change, if disorder is realized by delta impurities obeying Poisson statistics (or magnetic impurities as well, when projecting on the lowest Landau level (LLL) [4]). The spectrum is then bounded from below, and instead of having a tail, it becomes singular at low energy. More precisely, depending on a parameter  $f = \rho/\rho_l$  which is the ratio between the density of impurities ( $\rho$ ) and the Landau degeneracy ( $\rho_l$ ),

the DOS takes the following asymptotic form [2],

$$\lambda\rho(E) \sim_{\omega \rightarrow +0} \begin{cases} (1-f)\delta(\omega) + A(f)\omega^{-f}, & (0 < f < 1) \\ 1/\omega(\ln[\omega/\alpha])^2, & (f = 1) \\ B(f)\omega^{f-2}, & (1 < f < 2) \\ \text{const.}, & (f = 2) \\ C(f)\omega^{f-2}, & (f > 2) \end{cases} \quad (1)$$

with  $\omega = f(E - \omega_c)/\lambda\rho$  ( $\lambda$  measures the strength of the delta potential). This behavior is very uncommon, and seems to be particular to the choice of short-range single impurity potential; for long range ones, the usual Lifshitz tail is recovered [5]. In the standard Lifshitz argument with no magnetic field, low energy states are localized in regions of space where impurities are absent. An empty region, of typical size  $\pi R^2$ , contains states with energy of the order of  $1/(\pi R^2)$ . For a Poisson distribution, the probability of not finding a single impurity in a volume  $\pi R^2$  is  $\exp(-\rho\pi R^2)$ . Identifying the energy to the inverse size of the empty region let to obtain the low energy behavior  $\exp(-\frac{\rho}{E})$ . This heuristic argument [6] has been later confirmed and extended by field-theoretic approach [7–10]. The question is whether it is possible to adapt this argument for the problem with strong magnetic field, in order to have a physical interpretation of the base of the spectrum, known to be constituted of localized states [11].

Let us consider the case where the density of impurities is less than the Landau degeneracy ( $f < 1$ ).

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The zero energy delta peak has a simple interpretation [12] and corresponds to the delocalized states which are expected at the center of each Landau band [13, 14]: states are indeed linear combinations of Landau states, which vanish at the position of the impurities. In a given volume  $V$ , the number of available Landau states is  $\rho_l V$ . The number of constraints imposed on the zero energy states is  $\rho V$ , the number of impurities. As a consequence, the corresponding subspace of states has the dimension  $(\rho_l - \rho)V$  (unless, as will be seen later, two impurities coincide). This gives as expected the degeneracy  $\rho_l(1-f)$  per unit volume, given by (1). What remains to be analyzed is the  $\omega^{-f}$  behavior of the excited states spectrum.

The paper is organized as follows, in Section 1, the problem with a finite number  $N$  of impurities is analyzed in details. The zero modes are first extracted from the Hilbert space. This allows to define the restriction of the Hamiltonian to the excited subspace as a  $N \times N$  matrix. The two impurities case is explicitly solved: it exhibits the mechanism which produces low energy states. The generalization to a cluster of impurities is then considered and an estimation of the lowest energy is found. In Section 2, a statistical analysis is performed, using the latter estimation. The most probable configurations corresponding to a given low energy is found, and the contribution to the DOS is computed in the case  $f < 1$ .

## 2 The N delta impurities problem

### 2.1 Coherent states basis for the excited subspace

The  $N$  impurities problem, projected onto the LLL is defined by the Hamiltonian

$$H = \lambda P_0 \sum_{i=1}^N \delta(\mathbf{r} - \mathbf{r}_i) P_0 \quad (2)$$

after shifting the spectrum by a constant.  $\lambda$  is the coupling constant of the delta potential,  $P_0$  is the projection operator on the LLL. The basis corresponding to the symmetric gauge, centered at position  $a$  (using complex notation and magnetic units) writes:

$$\phi_p^a(\mathbf{r}) = \frac{1}{\sqrt{\pi p!}} (z - a)^p \exp \left[ -\frac{1}{2} (z\bar{z} + a\bar{a} - 2z\bar{a}) \right] p \in N. \quad (3)$$

When there is only one impurity located at position  $a$ , these states remain eigenstates, with zero energy for  $p > 0$  and with energy  $\lambda/\pi$  for  $p = 0$ . We associate the coherent state  $\psi_i$  to the impurity  $i$ , where  $\psi_i$  corresponds to the only non-vanishing state at  $\mathbf{r}_i$ :

$$\psi_i(\mathbf{r}) = \phi_0^{z_i}(\mathbf{r}) = \frac{1}{\sqrt{\pi}} \exp \left[ -\frac{1}{2} (z\bar{z} + z_i\bar{z}_i - 2z\bar{z}_i) \right]. \quad (4)$$

As already mentioned, the LLL is divided into two orthogonal subspaces: the zero energy subspace of dimension higher or equal to  $\rho_l V - N$ , and the excited subspace

of dimension less or equal to  $N$ . The subspace of wavefunctions vanishing at  $\mathbf{r}_i$ , is orthogonal to  $\psi_i$  and contains the zero energy states. Therefore the zero-energy subspace is orthogonal to the one generated by  $\psi_1, \dots, \psi_N$ . We now find under which conditions these states are linearly independent.

Given

$$\psi(\mathbf{r}) = \sum_{i=1}^N a_i \psi_i(\mathbf{r}) \quad (5)$$

a linear combination of these  $N$  states. In the Landau symmetric basis  $\psi$  writes:

$$\psi(\mathbf{r}) = \sum_{p=0}^{\infty} b_p \phi_p^0(\mathbf{r}). \quad (6)$$

The relation between the  $b_p$  and  $a_i$  is then

$$b_p = \frac{1}{\sqrt{p!}} \sum_{i=1}^N a_i \bar{z}_i^p e^{-\frac{1}{2} z_i \bar{z}_i}. \quad (7)$$

In order for  $\psi$  to be identically zero, the  $b_p$  have to vanish. Imposing this condition to the first  $N$  ( $p = 0 \dots N-1$ ) leads to an homogeneous system of equations for the  $a_n$ , with a determinant proportional to the  $\bar{z}_i$ 's Vandermonde determinant, *i.e.* a completely antisymmetric function of these variables. Therefore a necessary condition for the  $\psi_i$  to be linearly dependent is that two impurities coincide, and it is evidently sufficient. As a consequence  $\psi_1, \dots, \psi_N$  is a basis of the excited subspace (non-orthogonal).

We rewrite the Hamiltonian into this basis. Starting from the decomposition (5) of an arbitrary excited state, the action of  $H$  on this state is

$$\langle \mathbf{r} | H | \psi \rangle = \lambda \sum_{i=1}^N P_0(\mathbf{r}, \mathbf{r}_i) \sum_{n=1}^N a_n \psi_n(\mathbf{r}_i) \quad (8)$$

with

$$P_0(\mathbf{r}, \mathbf{r}') = \frac{1}{\pi} e^{-\frac{1}{2} (z\bar{z} + z'\bar{z}' - 2z\bar{z}')} \quad (9)$$

the kernel of the LLL projection operator. Using the fact that

$$\langle \psi_i | \psi_j \rangle = \pi P_0(\mathbf{r}_i, \mathbf{r}_j) = \sqrt{\pi} \psi_j(\mathbf{r}_i) \quad (10)$$

we obtain

$$\langle \mathbf{r} | H | \psi \rangle = \lambda \sum_{i=1}^N \sum_{j=1}^N a_j P_0(\mathbf{r}_i, \mathbf{r}_j) \psi_i(\mathbf{r}). \quad (11)$$

To conclude, the matrix elements of  $H$  expressed in the  $(\psi_1, \dots, \psi_N)$  basis are given by  $\lambda P_0(\mathbf{r}_i, \mathbf{r}_j)$ . The result is identical to the operator proposed in [15] for analyzing this problem.

## 2.2 Two impurities

With only two impurities, the matrix can be diagonalized. Choosing the spatial reference frame such that  $z_1 = -z_2 = a/2$ , where  $a$  is the distance between the two impurities, we have

$$H_2 = \frac{\lambda}{\pi} \begin{pmatrix} 1 & e^{-\frac{1}{2}a^2} \\ e^{-\frac{1}{2}a^2} & 1 \end{pmatrix}. \quad (12)$$

The eigenvalues of this matrix correspond to the energies  $E_-$  and  $E_+$  of the two excited states,

$$E_{\pm} = \frac{\lambda}{\pi} (1 \pm e^{-\frac{1}{2}a^2}), \quad (13)$$

and the corresponding wave-functions is (up to a normalization coefficient)

$$\psi_{\pm} = \psi_1 \pm \psi_2 = \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}(z\bar{z}+a^2)} (e^{2az} \pm e^{-2az}). \quad (14)$$

In this light, the excited states have almost the same energy, comparable to the one impurity value, when the two impurities are well separated; whereas, a low energy state is obtained when the two impurities are close. For  $a \ll 1$  this energy behave like

$$E_- \simeq \frac{\lambda}{2\pi} a^2. \quad (15)$$

## 2.3 Impurity cluster

The previous example suggests that low-energy states are associated with regions of high concentrations of impurities. Indeed,  $N$  impurities involve  $N$  localized states  $\psi_i$  (which have a characteristic size  $1/\rho_l$ ). And low-lying states are expected to appear when the overlap between these states starts to be important. For instance, when  $N$  impurities are confined in a small volume ( $\pi R^2$ ), forming in this way a cluster, the  $N$  corresponding states overlap essentially with the  $N_l = \rho_l \pi R^2$  Landau states situated inside the disc (in the symmetric gauge, the states of the LLL are localized on a ring of radius  $\sqrt{l/\rho_l \pi}$ , where  $l$  is the angular momentum [17]). So if  $N > N_l$ , we expect to have  $N - N_l$  low-energy states. It seems then natural to consider such configurations to analyze the lower spectrum.

First, we estimate the lowest energy of such configuration, using the decompositions (5) and (6) of an excited state. A low energy state is supposed to avoid the impurities. We construct such a state by imposing the  $b_p$  to vanish until  $p = N - 2$  (included).  $\psi$  has then components only on the Landau states  $p > N - 2$ , at a distance from the center of the cluster greater or equal to  $\sqrt{(N - 1)/\pi \rho_l}$ . The state is then given by the  $a_n$ , solutions of the set of equations

$$b_p = 0 = \sum_{n=1}^N a_n e^{-\frac{1}{2}z_n \bar{z}_n} z_n^p \quad p = 0, \dots, N - 2, \quad (16)$$

whose solution, up to a proportionality constant is

$$a_n e^{-\frac{1}{2}z_n \bar{z}_n} = C_{N,n}, \quad (17)$$

where  $C_{N,n}$  is the cofactor of the element  $(N, n)$  in the Vandermonde type matrix:

$$D_N^p = \begin{pmatrix} 1 & \dots & \dots & 1 \\ z_1 & \dots & \dots & z_N \\ \vdots & \dots & \dots & \vdots \\ z_1^{N-2} & \dots & \dots & z_N^{N-2} \\ z_1^p & \dots & \dots & z_N^p \end{pmatrix}. \quad (18)$$

In particular, for  $p = 0 \dots N - 2$ ,

$$\det D_N^p = 0 = \sum_{n=1}^N C_{N,n} z_n^p \quad (19)$$

which is precisely what we want. Moreover the  $C_{N,n}$  are given by the expression

$$C_{N,n} = (-1)^{\frac{N(N-1)}{2}+n} \prod_{p < q, p, q \neq n} (z_p - z_q). \quad (20)$$

The matrix  $H_N = \lambda P_0(\mathbf{r}_i, \mathbf{r}_j)$ , written in  $\psi_1, \dots, \psi_N$  basis, is self-adjoint and positive, so its smallest eigenvalue  $E_0$  verify the inequality:

$$E_0 \leq \frac{(\psi | H_N \psi)}{(\psi | \psi)}, \quad (21)$$

with the norm defined by,

$$(\psi | \psi) = \sum_{n=1}^N \bar{a}_n a_n. \quad (22)$$

From this choice and for the considered state the inequality rewrites

$$E_0 \leq E = \frac{\lambda \sum_{n,m} \bar{C}_{N,n} C_{N,m} e^{z_m \bar{z}_n}}{\pi \sum_n |C_{N,n}|^2 e^{|z_n|^2}}, \quad (23)$$

and

$$E \leq \frac{\lambda \sum_{n,m} \bar{C}_{N,n} C_{N,m} e^{z_m \bar{z}_n}}{\pi \sum_n |C_{N,n}|^2}. \quad (24)$$

Expanding the exponential in the previous expression, we note that the first non-vanishing term corresponds to  $(z_m \bar{z}_n)^{N-1}/(N-1)!$ , because the determinant of  $D_N^p$  is zero for  $p < N - 1$ . In addition, since  $|z_n|^2 \leq N_l \leq N$ , the set of terms has a rapid decay, which allows to neglect the remainder of the expansion. We then obtain

$$E_0 \leq \frac{\lambda}{\pi} \frac{1}{(N-1)!} \frac{|D_{N-1}|^2}{\sum_{n=1}^N |C_{N,n}|^2}, \quad (25)$$

with  $D_{N-1} = (-1)^{N(N-1)/2} \prod_{p < q} (z_p - z_q)$ , the Vandermonde determinant of the  $z_n$  variables. If  $n^*$  labels

the impurity for which,  $\prod_{p \neq n} |z_n - z_p|^2$  is minimum, then we have the inequality

$$\sum_{n=1}^N |C_{N,n}|^2 \geq N |C_{N,n^*}|^2. \quad (26)$$

This leads to the desired approximate form for  $E_0$

$$E_0 \propto \frac{\lambda}{\pi} \frac{1}{N!} \min_p \prod_{n \neq p} |z_p - z_n|^2, \quad (27)$$

which coincides with expression (15) in the two impurities case.

### 3 Cluster thermodynamic

We now shall make use of the expression (27) to understand the low energy behavior of the spectrum obtained by Brézin, Gross, Itzykson ( $f < 1$ ). We start from the principle that each impurity in the system gives rise to an excited state whose energy depends on the configuration of the other impurities. If the concentration around one impurity is high, *i.e.* if the impurity is in a cluster, the corresponding energy is low and not affected by the impurities situated outside of the cluster (too far away for any overlap effect). We therefore associate a low-energy state to the presence of a cluster around an impurity, and by extension, introduce a density of states per impurity. Let  $X_i$  be a variable parameterizing the cluster configuration of the impurity  $i$ . Its contribution to the density of state per impurity is proportional to the probability  $P(X_i)$  of being realized

$$\rho_i(E) = \int DX_i P(X_i) \delta(E(X_i) - E). \quad (28)$$

So in average, the low-energy density of states by unit volume is proportional to  $\rho$  times the preceding expression. If we use now the expression (27) to evaluate the energy of the clusters, we note that to a given energy corresponds a statistical ensemble of clusters. Each cluster is defined by its volume  $N_l$ , its mean density  $\nu = N/N_l > 1 > f$ , and by the positions  $z_i$ ,  $i = 1 \dots N$ , of the impurities in the cluster. At very low energy, the clusters are expected to be macroscopic objects, and may be described by a finite number of macroscopic variables giving the density profile, in place of the microscopic degrees of freedom (namely the individual positions of impurities). For this purpose, we first determine the distribution of positions in a cluster of energy  $E$ , size  $N_l$  and mean density  $\nu$ . For a given configuration the energy is

$$E = \exp \left[ \sum_{n=1}^N \log |z_n|^2 - N \log N + N \right], \quad (29)$$

using the Stirling formula ( $N! \simeq N^N e^{-N}$ ) and with  $0 \leq |z_n|^2 \leq N_l$  ( $N_l = \pi \rho_l R^2$ ). We consider a subdivision of the cluster in  $M$  cells, corresponding to intervals

of the  $|z_n|^2$  equal to  $a = N_l/M$  (cells with identical area  $\pi R^2/M = a/\rho_l = \pi \delta r^2 = \delta |z|^2/\rho_l$ ). If  $n_p$  is the number of impurities in the cell  $p$ , then the probability associated to this configuration  $(n_1, \dots, n_M)$  of the cluster is

$$P(n_1, \dots, n_M, N) = \frac{N!}{n_1! \dots n_M!} \left( \frac{1}{M} \right)^N \frac{(f N_l)^N}{N!} e^{-f N_l}. \quad (30)$$

Since macroscopic objects are best described with the continuum limit, we define ( $x = |z|^2/N_l = \frac{p}{N_l} a$ )

$$\nu(x) dx = n_p = \nu(x) \frac{1}{M} \quad 1 \ll M \ll N. \quad (31)$$

The energy takes then the form

$$\log E = N_l \int_0^1 [\nu(x) \log x + \nu - \nu \log \nu] dx, \quad (32)$$

and, at leading contribution in  $N_l$ , the probability is

$$\log P = N_l \int_0^1 \left[ \nu(x) \left( 1 - \log \frac{\nu(x)}{f} \right) - f \right] dx, \quad (33)$$

with the constraint

$$\int_0^1 \nu(x) dx = \nu. \quad (34)$$

We now need to determine the configuration for which  $\log P$  is maximum at fixed  $E$ ,  $N_l$  and  $\nu$ . Using a Lagrange multiplier for the energy constraint we obtain the saddle point equation

$$\frac{\partial \log P}{\partial \nu(x)} - \alpha \frac{\partial \log E}{\partial \nu(x)} = 0. \quad (35)$$

The solution, with proper normalization, is

$$\nu(x) = \nu(1 - \alpha) \left( \frac{x}{N_l} \right)^{-\alpha}, \quad (36)$$

$\alpha$  being implicitly determined through the relation between  $\gamma = 1/(1 - \alpha)$  and the energy,

$$\log E = -N_l(\nu(\log \nu - 1) + \gamma \nu). \quad (37)$$

The probability becomes such that

$$\log P = \log E - N_l(f - \nu(1 + \log \gamma f)). \quad (38)$$

At a given energy, the possible configurations are parameterized by  $(N_l, \nu)$ . The saddle point is determined by the set of equations

$$\left( \frac{\partial \log P}{\partial N_l} \right)_{\nu, E} = 0, \quad (39)$$

$$\left( \frac{\partial \log P}{\partial \nu} \right)_{N_l, E} = 0. \quad (40)$$

Using (37), which determines  $\gamma$  these equations rewrites

$$\log \gamma f + \frac{1 - \log \nu}{\gamma} - \frac{f}{\nu} = 0, \quad (41)$$

$$\log \gamma f - \frac{1}{\gamma} \log \nu = 0. \quad (42)$$

So finally,  $\log P$  has its maximum (which can be verified by computing second derivatives) at energy  $E$  when  $\nu = 1$ ,  $N_l = -f \log E / (1 - f)$ , which corresponds to  $\gamma = 1/f$ . The other solution ( $\nu = f$  and  $\gamma = 1$ ) is also a local maximum, but outside the range of interest for the parameters. Hence, for this type of configurations (parameterized now only by  $X = N_l$ ), we have the relation

$$\log \frac{P}{E} = -f \log E. \quad (43)$$

And, using (28) (with the change of variable  $DX \propto dE/E$ ), we arrive at the expected low-energy behavior of the density of states

$$\rho(E) \propto E^{-f}. \quad (44)$$

Moreover, states contributing to this behavior are associated to the existence of impurity clusters of size  $N_l = -f \log E / (1 - f)$  and the shape

$$\nu(x) = f x^{f-1}. \quad (45)$$

In contrary to the Lifshitz argument, the low energy states are associated with regions of high impurity concentration around which they localize. Their characteristic size is  $\log 1/E$ , a rough indication of a logarithmic divergence of the localization length, at least when  $f \ll 1$ . This feature (absence of critical exponent) might be very particular to the zero-range nature of the impurity scattering potential, and is consistent with numerical analysis found in reference [16]. When  $f$  approaches 1, this picture might be modified by some ‘‘percolation’’ effect of the clusters. For  $f$  greater than 1 the argument developed in this paper

is not applicable to reproduce the low energy spectrum, but neither is the standard Lifshitz argument. This seems to indicate that states are not localized at the bottom of the spectrum in this case.

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